Non-invertible symmetry, and string tensions beyond N-ality

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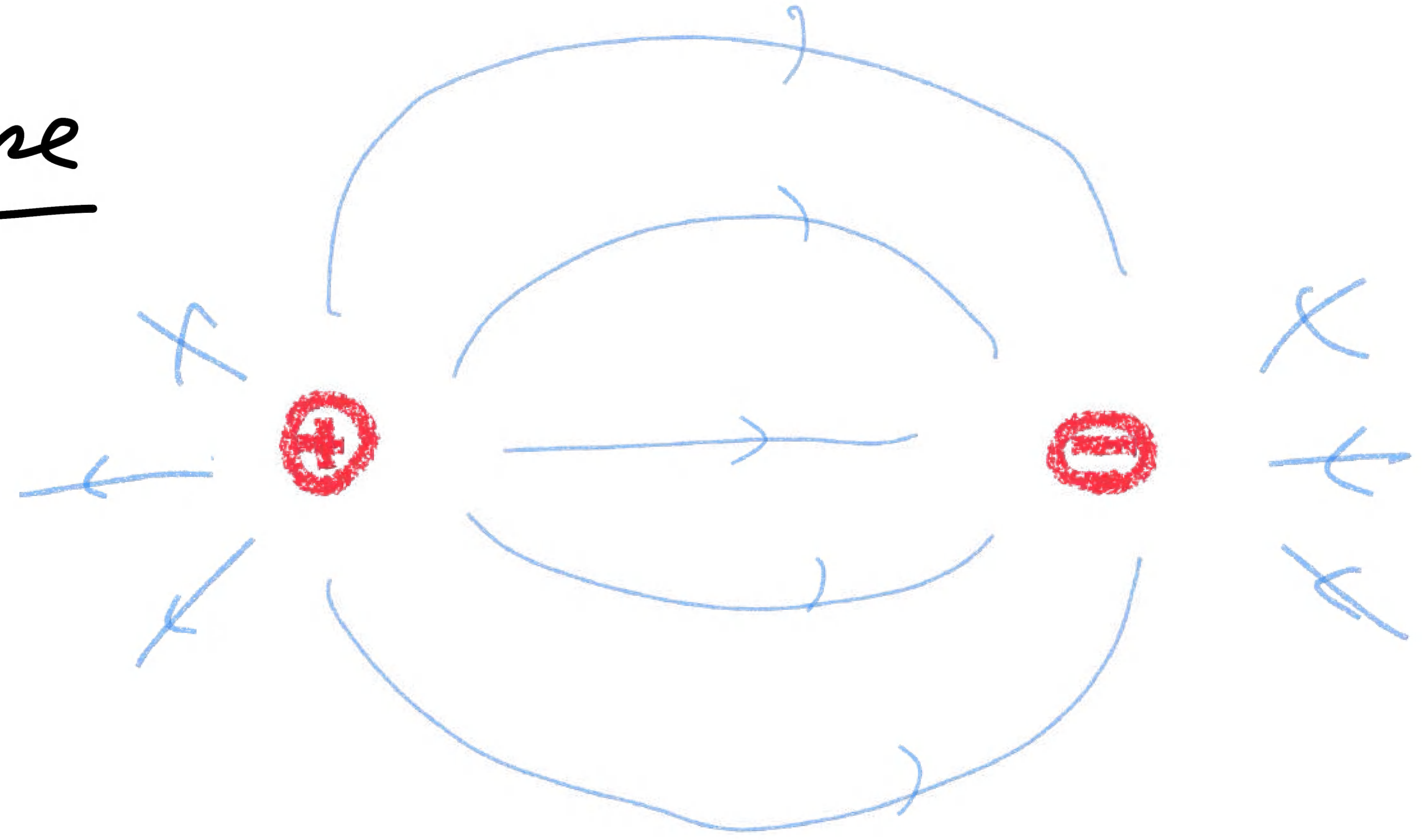
Motivation : Confining strings in SU(N) YM

In order to characterise "confinement" in gauge theories, we look at the interparticle potential for test charges.

Potentials

Coulomb phase

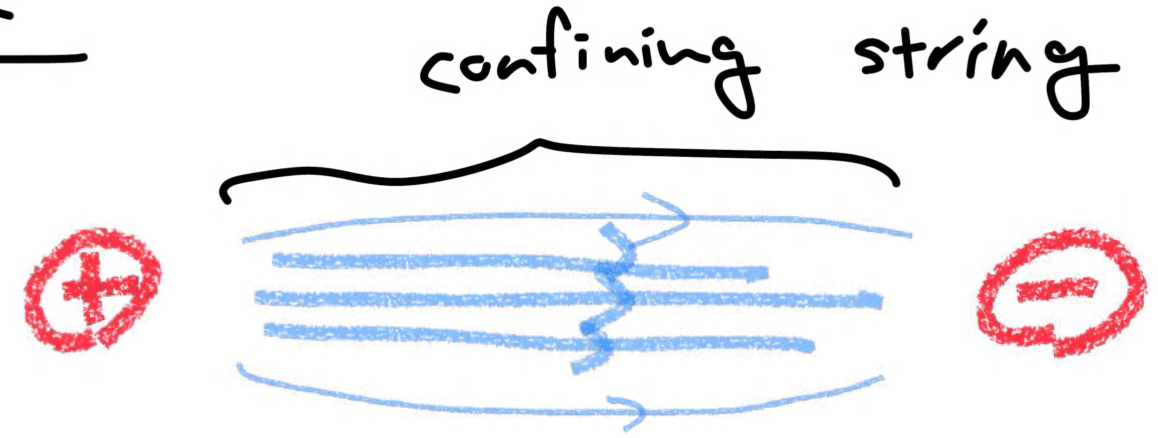


$$V(r) \propto \frac{1}{r^{D-2}} \qquad (D:\text{spatial dim.})$$

Higgs phase

$$V(r) \propto \text{const.}$$

Confining phase

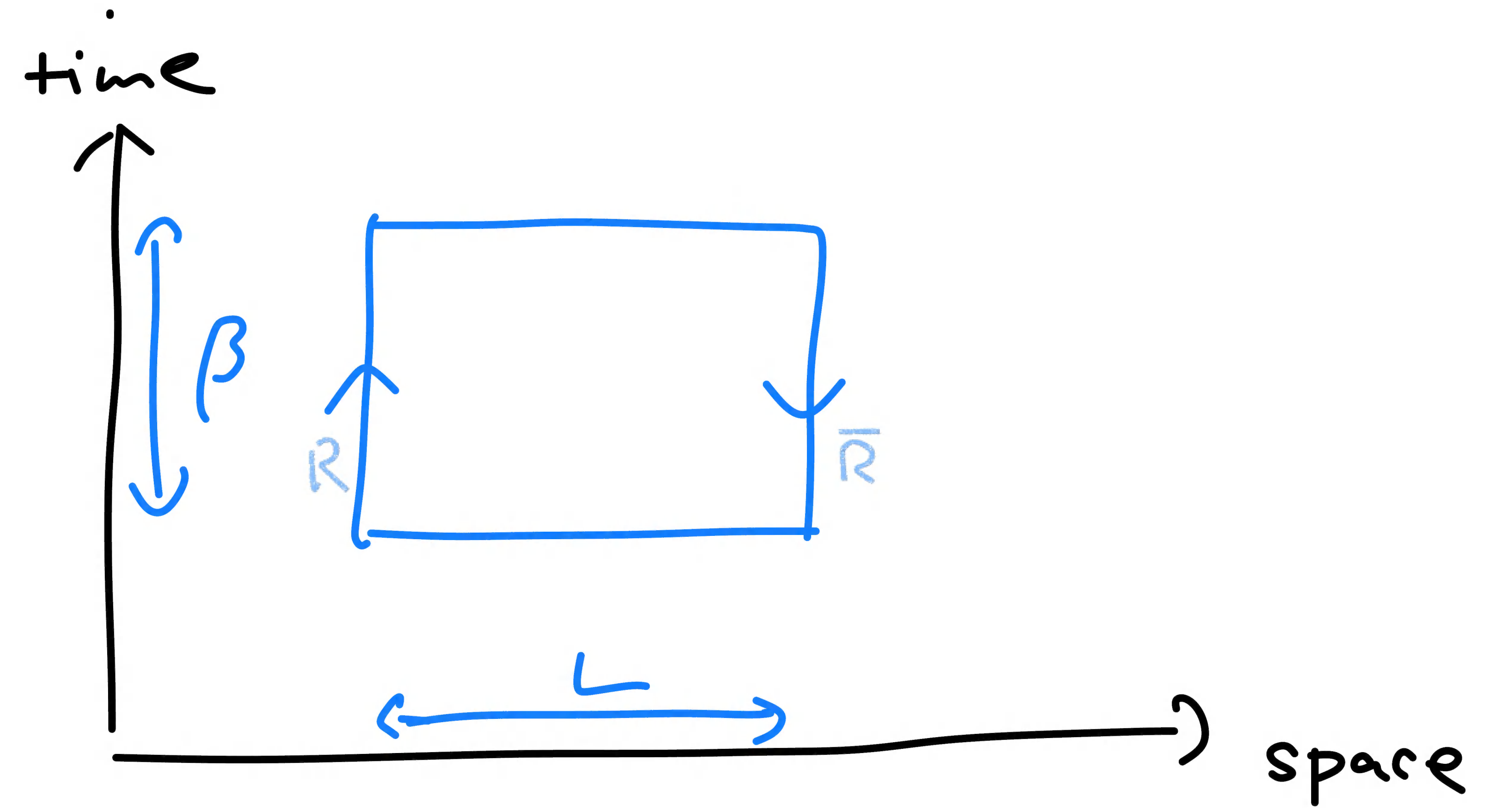


$$V(r) \propto r.$$

Wilson Loop & Area law

In order to detect the interparticle potential in QFT, we introduce a loop operator, such as the Wilson loop.

$$W_R(C) = \mathrm{tr}_R\left[\mathcal{P} e^{i\oint_C a_\mu dx^\mu} \right]$$



$$\rightsquigarrow \langle W_R(C) \rangle \sim \exp\left(-\beta\, V_R(L)\right),$$

where $V_R(L)$ is a potential for charges R, $\bar{R}$, separated by L.

Especially, when the system is confining

$$\langle W_R(C) \rangle \sim \exp\left(-\underset{\uparrow}{T_R} \times \mathrm{Area}(C)\right).$$

string tension

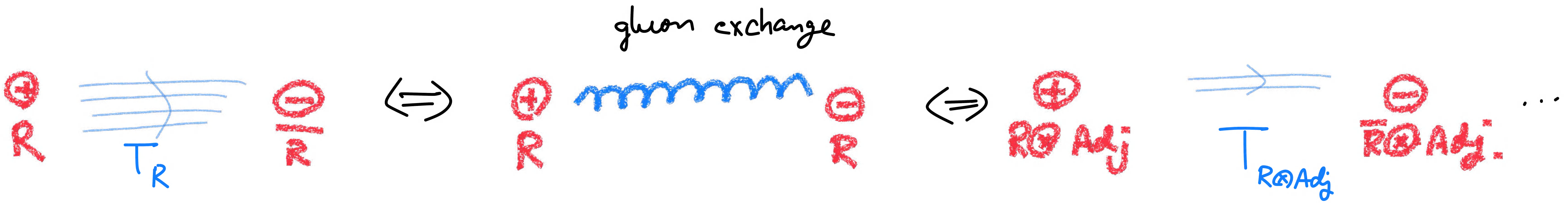
Spectrum of confining strings : N-ality

T_R (string tension) are important quantities characterizing confinement.

$\Rightarrow$ How do T_R's depend on the gauge representation R ?

N-ality T_R depends only on $\#$(boxes of Young tab.) mod N.

(Believed to be true for pure $SU(N)$ YM for 3 & 4 dim.)



1-form symmetry (center symmetry)

In modern understandings, for relativistic QFTs, (cf. Gaiotto, Kapustin, Seiberg, Willet '14)

Symmetry $\overset{\text{def}}{:=}$ existence of "topological" operators $U(M_{d-p-1})$.

Basic example (Usual sym)

Continuous sym. of $\mathcal{L}$ $\longrightarrow$ Noether current $j^\mu_\alpha = \delta_\alpha \phi \dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}$.

$$\Rightarrow \quad U_\varepsilon(M_{d-1}) := \exp\left(i \int_{M_{d-1}} \varepsilon_\alpha \, j^{\mu_1}_\alpha \, \varepsilon_{\mu_1 \mu_2 \cdots \mu_d} \, dx^{\mu_2} \cdots dx^{\mu_d} \right).$$

$\mathbb{Z}_N$ 1-form sym.

For $SU(N)$ YM, there are codim-2 topological operator $U(M_{d-2})$.

[Gukov-Witten operator]

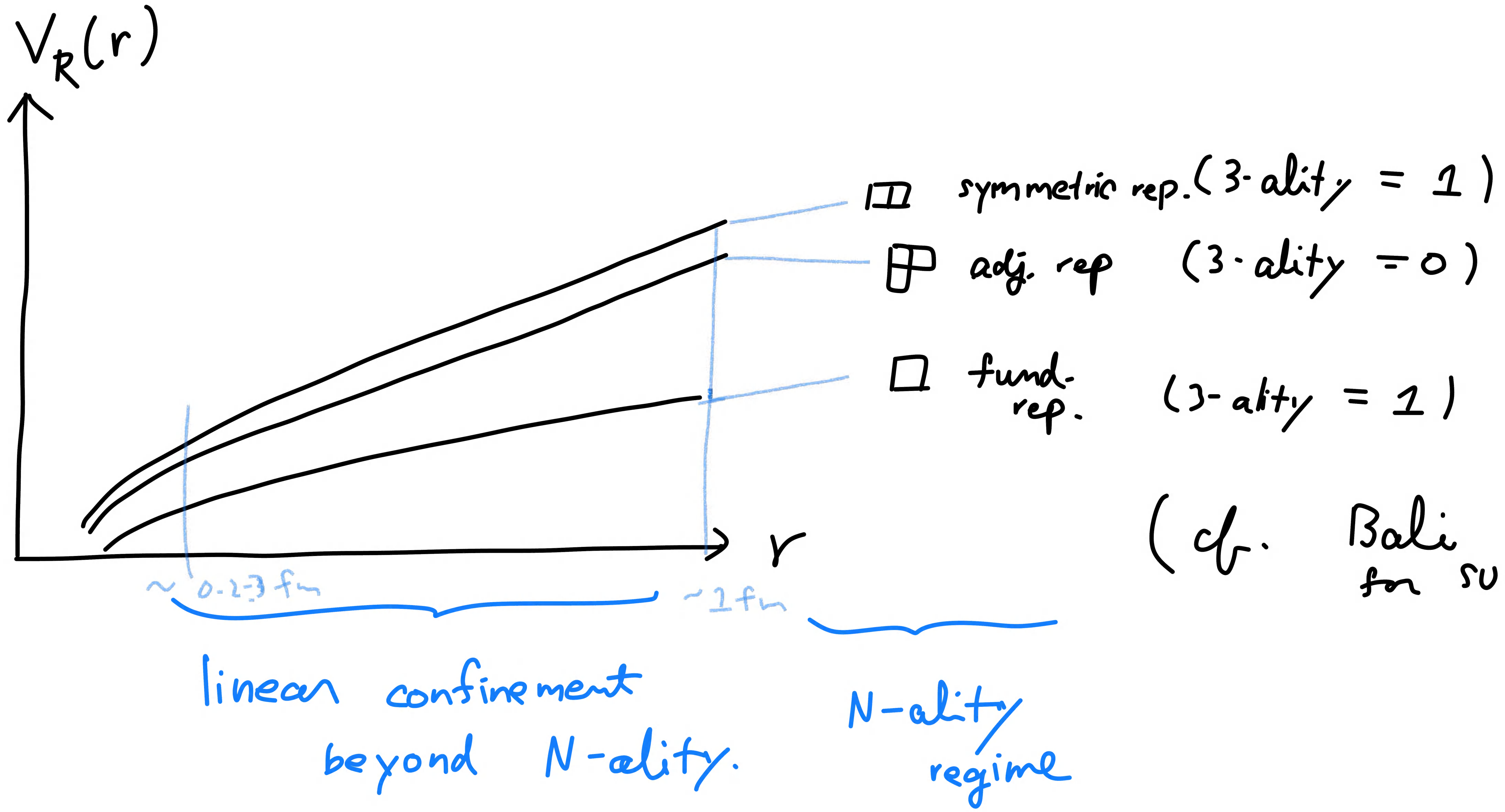
$$W_R(C) \longmapsto e^{\frac{2\pi i}{N} \cdot |R|} W_R(C).$$

$\Rightarrow$ Nice explanation of N-ality rule.

Beyond N-ality

N-ality is an important feature in the deep IR regime.

However, it does not kick in immediately after the linear confinement occurs.



(cf. Bali 2000 for SU(3) YM)

Is there some "nice" way to understand the beyond N-ality regime?

What is this talk about?

We want to get better understandings on confining strings beyond N-ality.

① Construction of a toy model (3d semi-Abelian gauge theory)

- $G_{\text{gauge}} = U(1)^{N-1} \rtimes S_N$
- $\mathbb{Z}_N$ 1-form sym.

② Studying its string tensions (string tensions beyond N-ality)

- $T_{\text{Adj}} \simeq 2\,T_{\text{fd.}} \neq 0$

③ "Symmetry" explanation on beyond N-ality (non-invertible sym. in 3d)

* Disclaimer : This talk does not contain SU(N) YM at all,
but it's anyway interesting!!

(2+1)d $U(1)^{N-1}$ gauge model with S_N global sym

Gauge field $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{pmatrix}$ with $\mathrm{tr}(\vec{a}) = \sum_{i=1}^{N} a_i = 0$.

(Using simple roots α_i of $SU(N)$, we can rewrite it as

$$\vec{a}(x) = \sum_{i=1}^{N-1} \tilde{a}_i(x) \alpha_i .$$
)

The Lagrangian

$$\mathcal{L} = \frac{1}{g^2} \vec{f}_{\mu\nu} \cdot \vec{f}_{\mu\nu}$$

is symmetric under the global S_N permutation

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{pmatrix} \longmapsto \begin{pmatrix} a_{\sigma(1)} \\ a_{\sigma(2)} \\ \vdots \\ a_{\sigma(N)} \end{pmatrix} \qquad (\sigma \in S_N).$$

(* Here, I write down the continuum Lagrangian,

but, in the actual computations, we used the lattice model.)

Monopoles

This (lattice) model acquires the mass gap due to monopole gas.

As we put the theory on lattice, Bianchi id.

$$df = d(da) = 0$$

is **violated: $df \neq 0$.** (lattice monopoles)

Abelian duality : $* \vec{f} = d\vec{\sigma}$ w/ $2\pi\mu$ - periodic scalars $\vec{\sigma}$.

Monopoles can be described as $e^{i\alpha\cdot\vec{\sigma}(x)}$.

$$\mathcal{L}_{\text{eff}} = \frac{g^2}{2\pi}\int \left\{ |d\vec{\sigma}|^2 + \underbrace{e^{-\frac{\#}{g^2}} \cdot \sum_{\alpha:\,\text{positive roots}} \left(1-\cos(\alpha\cdot\vec{\sigma})\right)}_{\Rightarrow\ \text{mass gap}} \right\} d^3x$$

Some remarks

Idea is very similar to the Polyakov model :

3d SU(N) YM + Adj. scalar Φ

$$\xrightarrow{\text{Higgsing by } \langle\Phi\rangle} \text{3d } U(1)^{N-1} \text{ gauge theory + monopoles.}$$

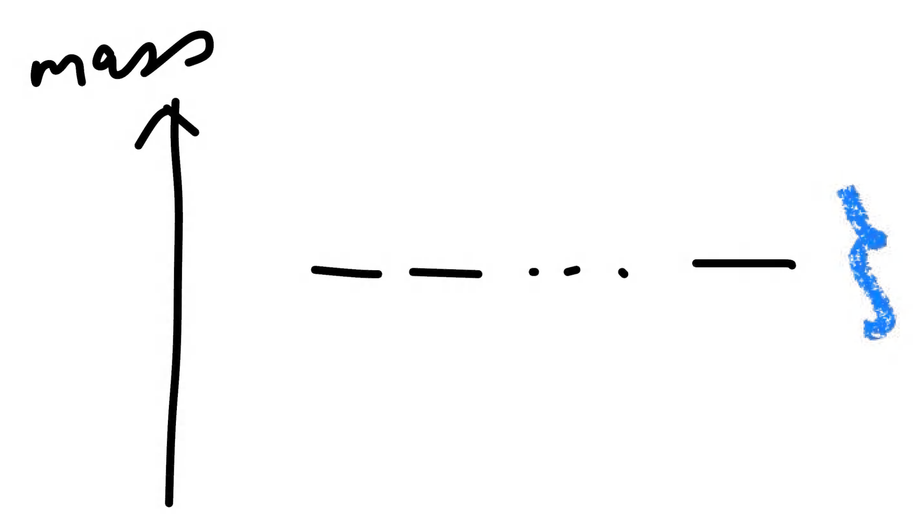
$$\mathcal{L}_{eff} \sim |d\vec{\sigma}|^2 + e^{-\frac{\#}{g^2}} \sum_{\alpha} \left(1-\cos(\alpha\cdot\vec{\sigma})\right)$$

α : positive simple roots

The Polyakov model, however, does not realize S_N symmetry at low-energies, while our model does.

Our model

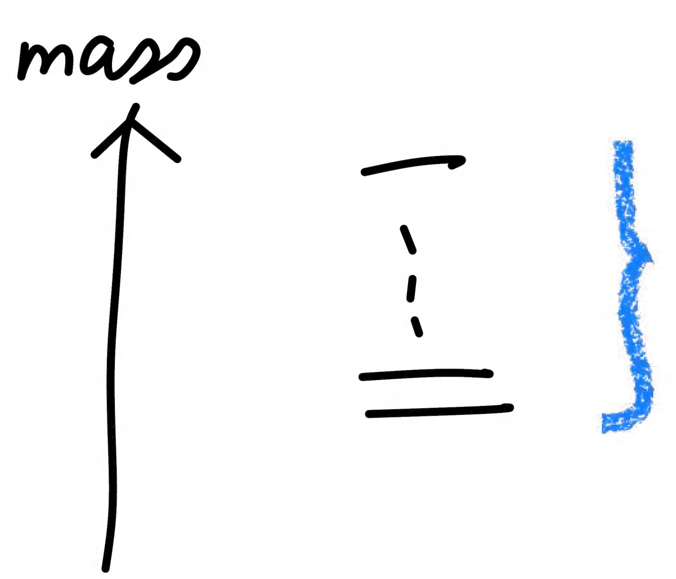
S_N ✓



(N-1) dual photons have degenerate mass

Polyakov model

S_N ✗



(N-1) dual photons have different masses.

Gauging S_N : $U(1)^{N-1} \rtimes S_N$ gauge theory

As our $U(1)^{N-1}$ model has a manifest S_N symmetry, let us gauge it!

$\Rightarrow$ Physical operators must be local S_N invariant.

Wilson formulation

On the lattice, we can realize it by saying that

$$\overset{\text{link variable}}{U_\ell} = \underbrace{P_\ell}_{\substack{N\times N \text{ matrix} \\ \text{for } S_N \text{ perm.}}} \cdot \underbrace{C_\ell}_{= \begin{pmatrix} e^{ia_1} & & \\ & \ddots & \\ & & e^{ia_N} \end{pmatrix} \in U(1)^{N-1}}$$

$$S = \underbrace{\beta_1 \, \mathrm{tr}\Big(\mathbb{1}_N - \prod_{\ell \in \partial p} (P_\ell \cdot C_\ell)\Big)}_{U(1)^{N-1} \rtimes S_N \text{ plaquette action}} + \underbrace{\beta_2 \, \mathrm{tr}\Big(\mathbb{1}_N - \prod_{\ell \in \partial p} P_\ell\Big)}_{S_N \text{ plaquette action}}.$$

Taking the limit $\beta_2 \to +\infty$, we impose the flatness condition

$$\prod_{\ell \in \partial p} P_\ell = \mathbb{1}_N .$$

$\Rightarrow$ S_N-gauge fields are topological.

In this limit, the local dynamics does not change under the gauging of S_N:

S_N Wilson line

$$\langle \mathcal{O}_1(x_1) \, W_{S_N}(x_1, x_2) \, \mathcal{O}_2(x_2) \rangle_{U(1)^{N-1} \rtimes S_N}$$

S_N-non-singlet operators.

$$= \langle \mathcal{O}_1(x_1) \, \mathcal{O}_2(x_2) \rangle_{U(1)^{N-1}} .$$

- Thanks to this nature, we can peform explicit computations as the theory is essentially Abelian.
- Symmetry is affected, however.

1-form symmetries of semi-Abelian theory

Gauge invariance

$$U(1)^{N-1} \xRightarrow{S_N\text{-gauging}} U(1)^{N-1} \rtimes S_N$$

"semi-Abelian"

$(\subset SU(N)$: subgrp of $SU(N))$

Center of the gauge group

$$Z(U(1)^{N-1}) = U(1)^{N-1}$$

$$\Downarrow$$

$U(1)^{N-1}$ 1-form sym.

$\rightsquigarrow$ Infinitely many string tensions can be expected.

$$Z(U(1)^{N-1} \rtimes S_N) = \mathbb{Z}_N.$$

$$\Downarrow$$

$$\boxed{\mathbb{Z}_N \text{ 1-form sym.}}$$

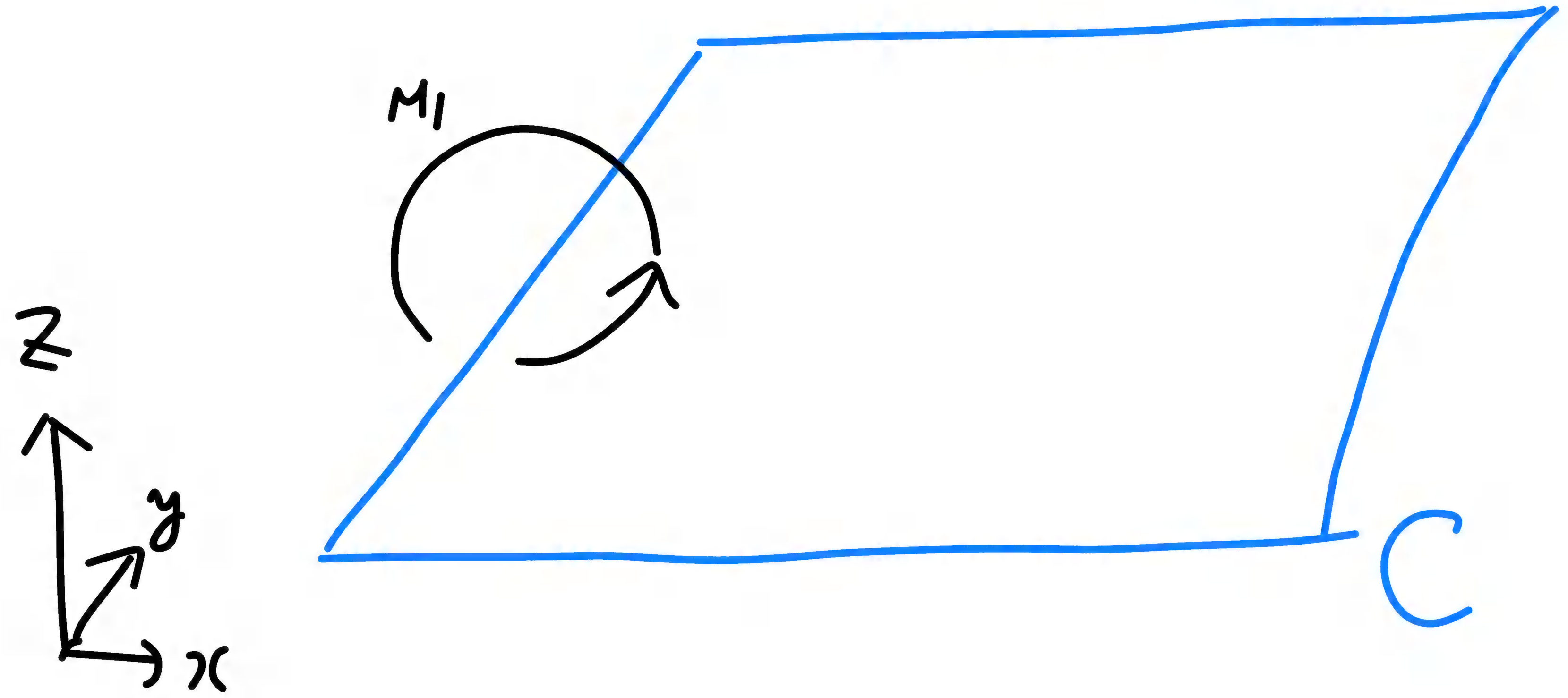
$\rightsquigarrow$ Only N-distinct strings can be naturally explained.

String tensions

In order to compute string tensions, we must know how the Wilson loops can be realized in the monopole theory

$$\mathcal{L}_{eff} = |d\vec{\sigma}|^2 + e^{-\frac{\#}{g^2}} \sum_{\alpha} \left(1-\cos(\alpha\cdot\vec{\sigma})\right).$$

<u>Defect operator</u>



$$\oint_{M_1} d\vec{\sigma} = 2\pi\vec{\mu} \qquad (\vec{\mu} \text{ is a weight vector})$$

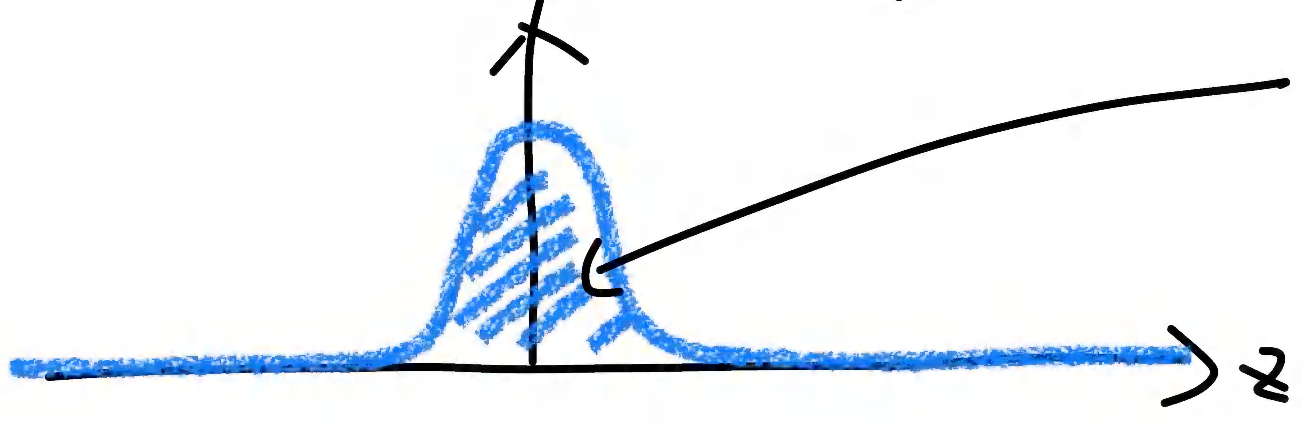
$$\Updownarrow$$

$$W_{\vec{\mu}}(C) = e^{i\vec{\mu}\cdot\oint_C \vec{a}}.$$

<u>σ- profile</u> Inside C, it looks like

<u>Energy density</u>

$\varepsilon(z) = \mathcal{L}_{eff}(\sigma(z))$



string tension $T_{\vec{\mu}}$.

String tensions beyond N-ality

Under a reasonable anatz, we obtain the string tensions for various charges:

$\underline{\mu_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} - \frac{1}{N}\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}}$ (SU(N) fundamental. N-ality $= 1$)

$$T_{\mu_1} \left(= \# e^{-\frac{\#}{g^2}} \times \frac{N-1}{\sqrt{N}} \right) \neq 0.$$

$\underline{\mu_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} - \frac{2}{N}\begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}}$ (SU(N) 2-index anti-sym. N-ality $=2$)

$$T_{\mu_2} = \frac{2(N-2)}{N-1} T_{\mu_1} \quad \left(< 2 T_{\mu_1} \right)$$

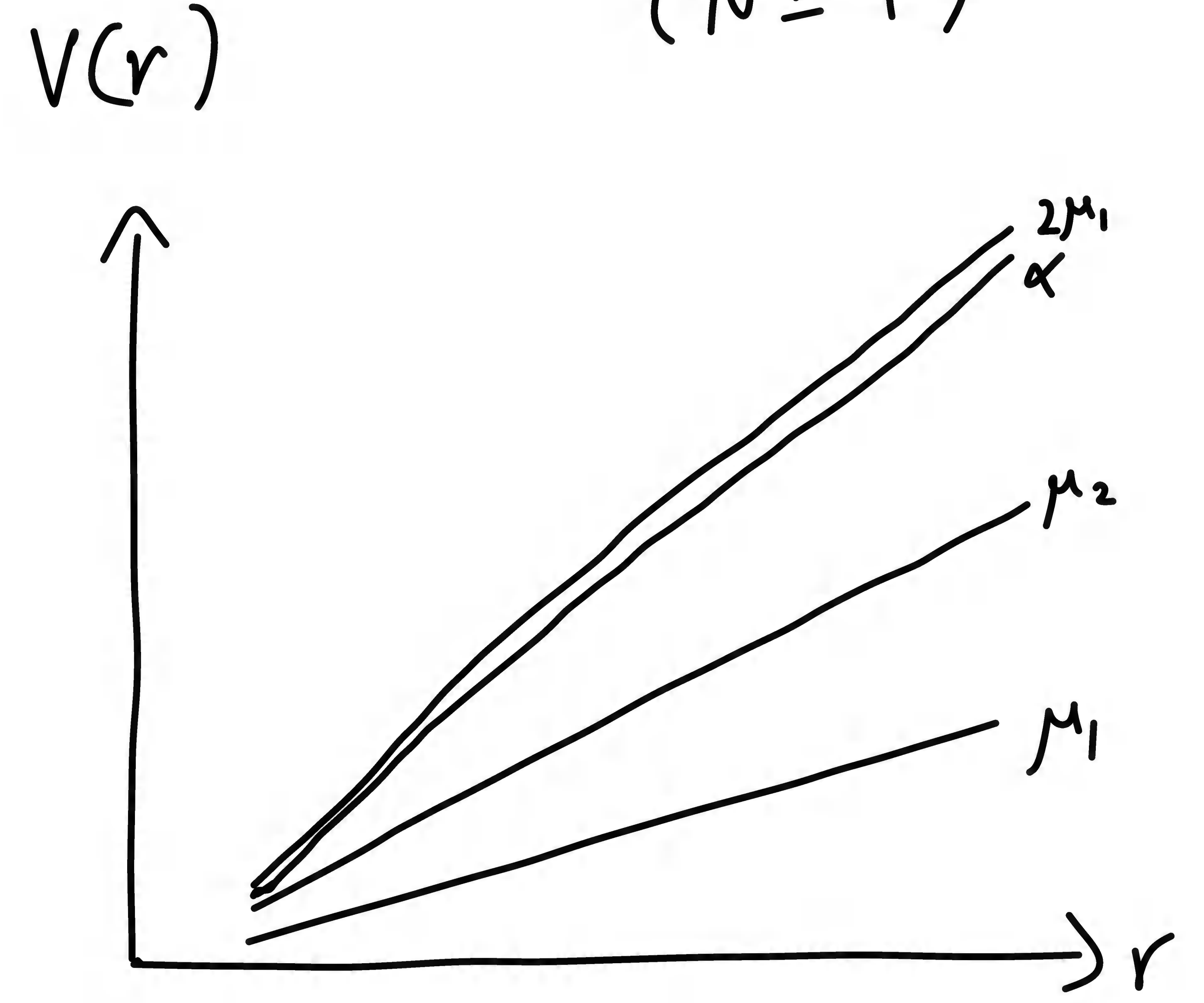
$\underline{2\mu_1}$ (SU(N) 2-index sym. N-ality $= 2$)

$$T_{2\mu_1} = 2 T_{\mu_1}$$

$\underline{\alpha_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}$ (SU(N) adjoint. N-ality $= 0$)

$$T_{\alpha} \simeq 2 T_{\mu_1}.$$

($N \geq 4$)



Natural or unnatural ?

For $U(1)^{N-1}$ model, this is natural. We have $U(1)^{N-1}$ 1-form sym. generated by

$$U_\theta^{(k)}(M_1) = \exp\left(i \frac{\theta}{2\pi} \oint_{M_1} \alpha_k \cdot d\vec{\sigma} \right) \qquad (k = 1, \cdots, N-1).$$

$\Rightarrow$ Infinitely many strings can be explained by this symmetry.

For $U(1)^{N-1} \rtimes S_N$ model, these operators are not gauge invariant.

We must consider a special combination

$$U_n(M_1) = \prod_{k=1}^{N-1} U^{(k)}_{\frac{2\pi}{N} nk}(M_1)$$

$$= \exp\left(i \frac{n}{N} \oint_{M_1} (\alpha_1 + 2\alpha_2 + \cdots + (N-1)\alpha_{N-1}) \cdot d\sigma \right).$$

This obeys the $\mathbb{Z}_N$ multiplication law :

$$U_n U_m = U_{n+m \bmod N}.$$

$\Rightarrow$ Only N distinct strings can be expected. Unnatural situation ??

Non-invertible symmetry

We can construct another class of S_N-inv. operators from $U_\alpha^{(k)}(M_1)$:

$$\mathcal{J}_\theta(M_1) \equiv \frac{1}{N!} \sum_{P\in S_N} P\, U_\theta^{(1)}(M_1)\, P^{-1}$$

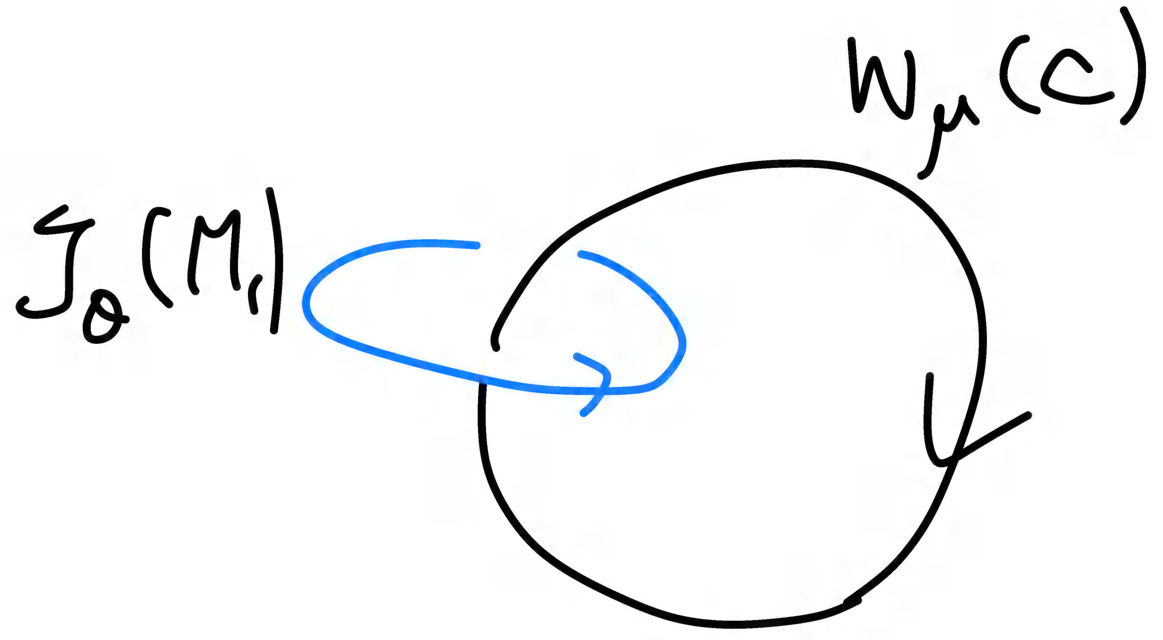
$$= \frac{1}{N(N-1)} \sum_{\alpha:\text{ roots}} \exp\left(i \frac{\theta}{2\pi} \oint_{M_1} \alpha\cdot d\sigma\right).$$

- $\mathcal{J}_\theta(M_1)$ is topological $\Rightarrow$ "Symmetry"
- $\mathcal{J}_\theta(M_1)$ does not form a group. $\mathcal{J}_\theta \cdot \mathcal{J}_{\theta'} \neq \mathcal{J}_{\theta+\theta'}$
- Moreover, $\mathcal{J}_\theta(M_1)$ does not correspond to unitary/anti-unitary operations.

Even though the last two properties are very unusual as symmetries, we regard $\mathcal{J}_\theta$ as generators of "non-invertible symmetry".

(cf. Bhardwaj, Tachikawa '17, Buican, Gromov, '17, Thorngren, Wang '19, Komargodski, Ohmori, Roumpedakis, Seifnashri '20 etc.)
(for 2d QFTs)

Transformation by $\mathcal{J}_\theta$



$$= \left(\frac{1}{N(N-1)} \sum_\alpha e^{i\theta \vec{\alpha}\cdot\vec{\mu}} \right) \times W_\mu(C)$$

$\mu = \mu_1$ (N-ality 1)

$$W_{fd.} \longmapsto \frac{N - 2(1-\cos(\theta))}{N} W_{fd.}$$

As $\mathcal{J}_\theta$ is not unitary, its "eigenvalue" is smaller than 1.
Moreover, it can be 0 in some cases. ("non-invertible")

$\mu = \alpha_1$ (Adjoint rep. N-ality 0)

$$W_{adj.} \longmapsto \frac{(N-2)(N-3) + 4(N-2)\cos(\theta) + 2\cos(2\theta)}{N(N-1)} W_{adj}.$$

$\rightsquigarrow$ $\mathcal{J}_\theta$ can distinguish the adj. rep. from the trivial rep.
Consistent with $T_\alpha \neq 0$.

N-ality 2 representations

As another excercise, we try to distinguish N-ality 2 representations, 2-index sym. ($\square\!\square$) and anti-sym ($\boxminus$) reps.

highest weight $= 2\mu_1$ highest weight μ_2

Eigenoperators of J_θ (M_1) turn out to be W_{asym} and $(W_{sym} - W_{asym})$.

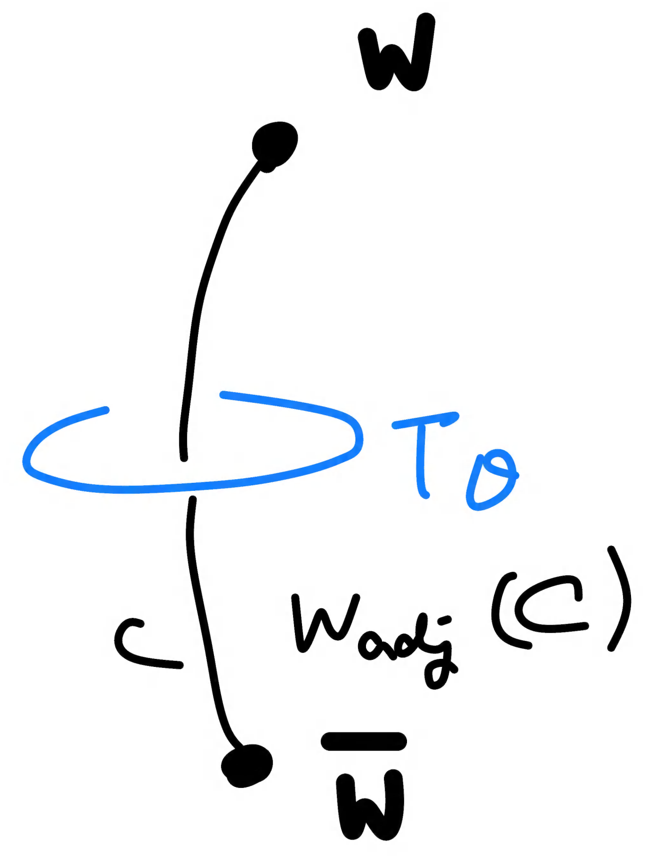
$$W_{sym} - W_{asym} \xmapsto{J_\theta} \frac{N - 2(1-\cos(2\theta))}{N} (W_{sym} - W_{asym})$$

$$W_{asym} \xmapsto{J_\theta} \frac{(N-2)(N-3) + 2 + 4(N-2)\cos\theta}{N(N-1)} \cdot W_{asym}.$$

Consistent with $T_{\mu_2} \neq T_{2\mu_1}$.

Effect of dynamical charges

Let us add dynamical electric matters with charges α's.
(W-bosons)



$\mathcal{T}_\theta$ is topological

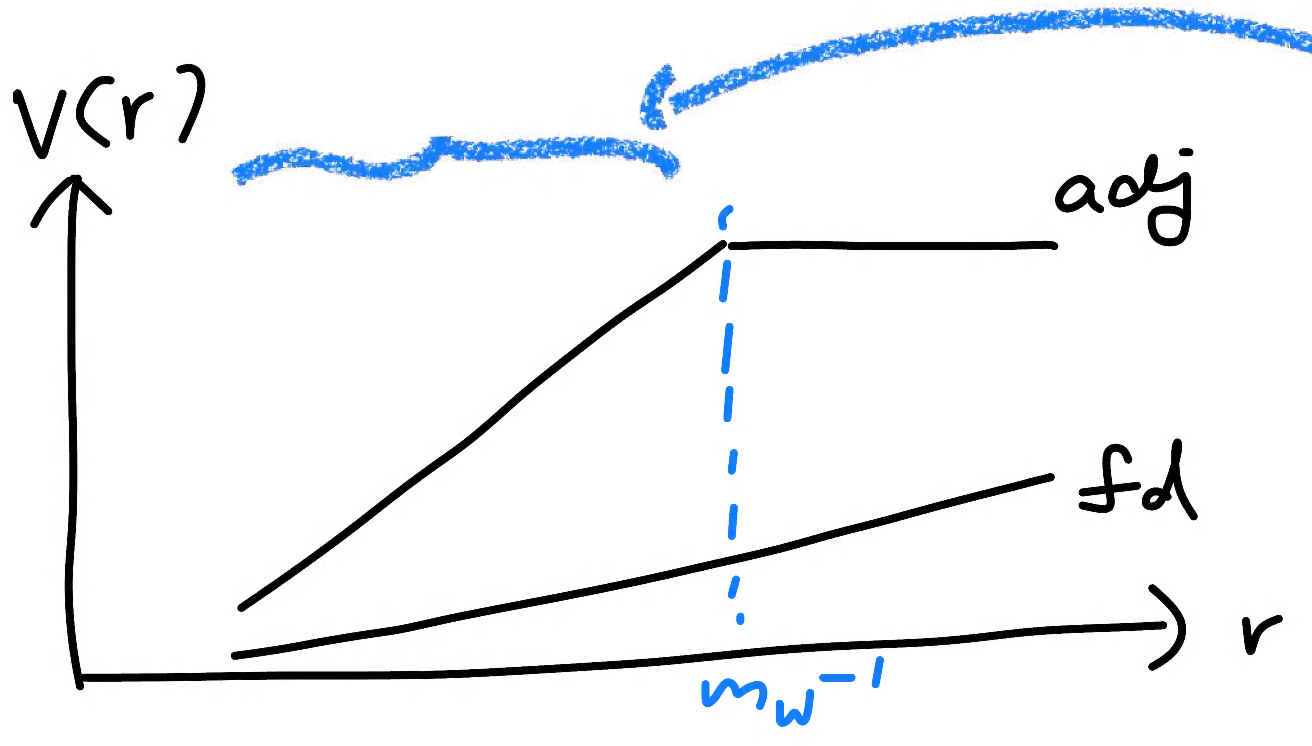
$$\Rightarrow \quad \mathcal{T}_\theta \cdot W_{adj} \overset{!}{=} W_{adj}.$$

$$\mathcal{T}_\theta \cdot W_{adj} = \frac{(N-2)(N-3)+4(N-2)\cos\theta+2\cos(2\theta)}{N(N-1)} W_{adj}$$

The equation holds only if $\underline{\theta=0}$ $(N\geq 3)$.

$\Rightarrow$ Non-invertible sym. is explicitly broken. (cf. Rudelius, Shao '20)

Still, if W's are heavy,



This regime can be understood by "softly-broken" non-invertible symmetry.

Speculations in Yang-Mills

In lattice simulation, the linear confinement is observed with the rep.-dep. string tensions.
(Casimir scaling, ...)

Also, in the large-N limit, factorization tells

$$\langle W_{adj}(C) \rangle \sim \langle W_{fd}(C) \rangle \cdot \langle W_{\overline{fd}}(C) \rangle$$

$$\Rightarrow \quad T_{adj} = 2\, T_{fd} \ (\neq 0).$$

It's an interesting question if these behaviors could be understandable by (approximate, or large-N emergent) non-invertible symmetries!!

Summary

① Construction of a toy model (3d semi-Abelian gauge theory)

- $G_{\text{gauge}} = U(1)^{N-1} \rtimes S_N$
- $\mathbb{Z}_N$ 1-form sym.

② Studying its string tensions (string tensions beyond N-ality)

- $T_{\text{Adj}} \simeq 2\,T_{\text{fd.}} \neq 0$
- $U_1(M_1) = \exp\left(i \frac{1}{N} \oint_{M_1} (\alpha_1 + 2\alpha_2 + \cdots + (N-1)\alpha_{N-1}) \cdot d\sigma\right)$ detects N-ality.

③ "Symmetry" explanation on beyond N-ality (non-invertible sym. in 3d)

- $J_\theta(M_1) = \frac{1}{N(N-1)} \sum_{\alpha:\text{roots}} \exp\left(i \frac{\theta}{2\pi} \int_{M_1} \alpha \cdot d\sigma\right)$

  is a topological, gauge-inv. operator.

- $J_\theta \cdot W_{\text{adj}} = (\theta\text{-dep. factor}) \times W_{\text{adj}}$.

  $\Rightarrow$ Trivial N-ality can have nonzero string tensions.

Applicable to SU(N) YM ?